Algebraic Properties of Curvature Operators in Lorentzian Manifolds with Large Isometry Groups

Giovanni CALVARUSO [†] *and Eduardo GARCÍA-RÍO* [‡]

[†] *Dipartimento di Matematica "E. De Giorgi", Università del Salento, Lecce, Italy*
  E-mail: *giovanni.calvaruso@unisalento.it*

[‡] *Faculty of Mathematics, University of Santiago de Compostela,*
  15782 Santiago de Compostela, Spain
  E-mail: *eduardo.garcia.rio@usc.es*

Abstract. Together with spaces of constant sectional curvature and products of a real line with a manifold of constant curvature, the socalled *Egorov spaces* and *ε-spaces* exhaust the class of n-dimensional Lorentzian manifolds admitting a group of isometries of dimension at least $\frac{1}{2}n(n-1)+1$, for almost all values of n [Patrangenaru V., *Geom. Dedicata* **102** (2003), 25–33]. We shall prove that the curvature tensor of these spaces satisfy several interesting algebraic properties. In particular, we will show that Egorov spaces are Ivanov–Petrova manifolds, curvature-Ricci commuting (indeed, semi-symmetric) and $\mathcal{P}$-spaces, and that ε-spaces are Ivanov–Petrova and curvature-curvature commuting manifolds.

Key words: Lorentzian manifolds; skew-symmetric curvature operator; Jacobi, Szabó and skew-symmetric curvature operators; commuting curvature operators; IP manifolds; $\mathcal{C}$-spaces and $\mathcal{P}$-spaces

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1 Introduction

The study of the relations between algebraic properties of the curvature tensor and the geometry of the underlying manifold, is a field of great interest, which has been intensively studied in recent years. In general, one finds different behaviours according to different possibilities for the signature of the metric, and the Riemannian case usually turns out to be much more restrictive than the Lorentzian and the pseudo-Riemannian ones. In this paper, we shall emphasize some interesting properties, determined by the curvature tensor, for two remarkable classes of n-dimensional Lorentzian manifolds, known as *Egorov spaces* and *ε-spaces*, which naturally occur in the classification of Lorentzian manifolds admitting a group of isometries of dimension at least $\frac{1}{2}n(n-1)+1$.

These spaces do not have a Riemannian counterpart. In fact, an n-dimensional Riemannian manifold that admits a group of isometries of dimension $\frac{1}{2}n(n-1)+1$, is either of constant sectional curvature or the Riemannian product between an $(n-1)$-dimensional manifold of constant sectional curvature and a line (or circle). More cases are possible in Lorentzian settings, because of the existence of null submanifolds which are left invariant by group actions.

Let $l_0(n) > l_1(n) > \cdots$ denote the possible dimensions of all groups of isometries of Lorentzian manifolds of dimension n. Following [21], an n-dimensional connected Lorentzian manifold M is said to belong to the *j-stratum* if there is a Lie group G, of dimension $l_j(n)$, that acts effectively on M by isometries. The third stratum is formed by Lorentzian manifolds admitting a group of isometries of dimension at least $\frac{1}{2}n(n-1)+1$.

Lorentzian manifolds belonging to the first two strata (that is, admitting a group of isometries of dimension at least $\frac{1}{2}n(n-1)+2$) have constant curvature. The complete classification of Lorentzian manifolds in the third stratum was also given in [21], in any dimension n greater than 5 and different from 7. Besides Lorentzian manifolds of constant curvature $M_1^n(c)$ and manifolds reducible as products $M^{n-1}(c) \times \mathbb{R}_1$ and $\mathbb{R} \times M_1^{n-1}(c)$, the remaining examples are:

- ε-*spaces:* Lorentzian manifolds $(\mathbb{R}^n, g_\varepsilon)$, where $\varepsilon = \pm 1$ and

$$g_\varepsilon = \sum_{i=1}^{n-2}(dx_i)^2 - dx_{n-1}dx_n + \varepsilon \left(\sum_{i=1}^{n-2} x_i^2 \right)(dx_{n-1})^2. \tag{1}$$

These spaces are irreducible Lorentzian symmetric spaces which are models for non-symmetric spaces [7, 22].

- *Egorov spaces:* Lorentzian manifolds $(\mathbb{R}^n, g_f)$, where f is a positive smooth function of a real variable and

$$g_f = f(x_n) \sum_{i=1}^{n-2}(dx_i)^2 + 2dx_{n-1}dx_n. \tag{2}$$

These manifolds are named after I.P. Egorov, who first introduced and studied them in [8].

The paper is organized in the following way. The basic description of the curvature of Egorov spaces and ε-spaces, obtained in [1], will be reported in Section 2. In Sections 3, 4 and 5 we shall respectively deal with properties related to the skew-symmetric curvature operator, the Jacobi operator and the Szabó operator. Among other properties, it is remarkable that both Egorov spaces and ε-spaces are Ivanov–Petrova (Theorems 3 and 4), curvature-Ricci and curvature-curvature commuting as Theorems 6 and 8 show. It is worth to point out that Ricci operators are two-step nilpotent in both cases, as opposed to the case of simple curvature-Ricci commuting models considered in [15]. In Section 6 we shall prove that Egorov spaces are of recurrent curvature. As a consequence, such spaces are also $\mathfrak{P}$-spaces. Hence, under this point of view, they are as close as possible to being locally symmetric, even if most of them are not even locally homogeneous [1]. We shall conclude with some general results on Lorentzian manifolds in the third stratum.

2   On the curvature of Egorov spaces and ε-spaces

In [1], the first author, W. Batat and B. De Leo described the curvature and the Ricci tensor of Egorov spaces, investigating some curvature properties as in particular local symmetry, local homogeneity and conformal flatness.

Let $(\mathbb{R}^n, g_f)$, $n \geq 3$ denote an Egorov space. As proved in [1], with respect to the basis of coordinate vector fields $\{\partial_i = \frac{\partial}{\partial x_i}\}$ for which (2) holds, the possibly non-vanishing covariant derivatives of coordinates vector fields are given by

$$\nabla_{\partial_i}\partial_i = -\frac{f'}{2}\partial_{n-1}, \qquad \nabla_{\partial_i}\partial_n = \frac{f'}{2f}\partial_i, \qquad i = 1, \ldots, n-2. \tag{3}$$

Moreover, the components of the $(0,4)$-curvature tensor, which is taken with the sign convention $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, and $R(X,Y,Z,T) = \langle R(X,Y)Z,T \rangle$ are given by

$$R_{inin} = \frac{1}{4f}\left[(f')^2 - 2ff''\right], \qquad i = 1, \ldots, n-2, \qquad R_{ijkh} = 0 \quad \text{otherwise} \tag{4}$$

and the components of the Ricci tensor, $\text{Ric}(X,Y) = \text{trace}\,\{Z \mapsto R(X,Z)Y\}$, with respect to $\{\partial_i\}$ are

$$\text{Ric}_{nn} = \frac{n-2}{4f^2}[(f')^2 - 2ff''], \qquad \text{Ric}_{ij} = 0 \quad \text{otherwise.} \tag{5}$$

Consequently, the *Ricci operator* $\widehat{\text{Ric}}$, given by $\langle \widehat{\text{Ric}}X, Y \rangle = \text{Ric}(X,Y)$, is described as follows:

$$\widehat{\text{Ric}} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{n-2}{4f^2}[(f')^2 - 2ff''] \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{6}$$

The following properties were proved in [1]:

Theorem 1 ([1]). *All Egorov spaces $(\mathbb{R}^n, g_f)$, $n \geq 3$,*

(i) *are geodesically complete;*

(ii) *admit a parallel null vector field ∂_{n-1};*

(iii) *have a two-step nilpotent Ricci operator;*

(iv) *are conformally flat.*

It is also worthwhile to remark that among all Lorentzian manifolds in the third stratum, Egorov spaces are the only ones which need not be symmetric (indeed, not even homogeneous). In fact, as proved in [1], an Egorov space $(\mathbb{R}^n, g_f)$ is locally symmetric if and only if its defining function f satisfies

$$(f')^2 - 2ff'' = kf^2,$$

where k is a real constant, and is locally homogeneous if and only if either it is locally symmetric, or its defining function f is a solution of

$$(f')^2 - 2ff'' = \frac{c_n}{(x_n + d)^2} f^2,$$

for some real constants $c_n \neq 0$ and d.

Further observe that, as a consequence of (3) the coordinate vector field ∂_{n-1} is parallel and null, thus showing that the underlying structure of Egorov spaces is that of a Walker metric [4].

Next, let $(\mathbb{R}^n, g_\varepsilon)$, $n \geq 3$, denote a ε-space as described by (1). Then, the Levi-Civita connection of $(\mathbb{R}^n, g_\varepsilon)$ is completely determined by

$$\nabla_{\partial_i}\partial_{n-1} = -2\varepsilon x_i \partial_n, \quad i = 1, \ldots, n-2, \qquad \nabla_{\partial_{n-1}}\partial_{n-1} = -\varepsilon \sum_{k=1}^{n-2} x_k \partial_k. \tag{7}$$

The curvature components with respect to the coordinate basis $\{\partial_i\}$ are given by

$$R_{in-1in-1} = -\varepsilon, \quad i = 1, \ldots, n-2, \qquad R_{ijkh} = 0 \quad \text{otherwise} \tag{8}$$

(see [1]), and the components of the Ricci tensor are

$$\text{Ric}_{n-1\,n-1} = -(n-2)\varepsilon, \qquad \text{Ric}_{ij} = 0 \quad \text{otherwise.}$$

Moreover, the following results hold:

Theorem 2 ([1, 7]). *All ε-spaces $(\mathbb{R}^n, g_\varepsilon)$, $n \geq 3$,*

   (i) *are locally symmetric;*

  (ii) *admit a parallel null vector field ∂_n;*

 (iii) *have a two-step nilpotent Ricci operator;*

 (iv) *are conformally flat.*

It immediately follows from (7) that ε-spaces are Walker manifolds (indeed pp-waves) with ∂_n a parallel null vector field. We refer to [18] and references therein for more information on the geometry of pp-waves.

3   The skew-symmetric curvature operator

Let (M, g) be a pseudo-Riemannian manifold and R its curvature tensor. The *skew-symmetric curvature operator* of an oriented non-degenerate 2-plane π of M, is defined as

$$R(\pi) = |g(u,u)g(v,v) - g(u,v)^2|^{-1/2} R(u,v)$$

and is independent of the oriented pair $\{u, v\}$ of tangent vectors spanning π. (M, g) is said to be an *Ivanov–Petrova manifold* (shortly, an *IP manifold*) if the eigenvalues of $R(\pi)$ are constant on the Grassmannian $G^+(2, n)$ of all oriented 2-planes. The eigenvalues may change at different points of (M, g). These manifolds are named after S. Ivanov and I. Petrova, who introduced them and first undertook their study [19].

Riemannian IP manifolds have been completely classified in all dimensions $n \geq 4$ (see [13, 11, 20]). Examples of three-dimensional Riemannian IP manifolds which are neither conformally flat nor curvature homogeneous may be found in [5] and [20].

Recently, IP manifolds have been extended and largely investigated in pseudo-Riemannian settings (see [9, 12, 16] and references therein).

Consider now an Egorov space $(\mathbb{R}^n, g_f)$. Let π denote any oriented non-degenerate 2-plane and $\{u, v\}$ an orthonormal basis of π and put $\Xi = |g(u,u)g(v,v) - g(u,v)^2|^{1/2}$. The metric components with respect to $\{\partial_i\}$ can be deduced at once from (2), and the curvature components are given by (4). Then, a direct calculation shows that the skew-symmetric curvature operator $R(\pi) = \frac{1}{\Xi} R(u,v)$ takes the form

$$R(\pi) = \frac{(f')^2 - 2ff''}{4\Xi f^2} \begin{pmatrix} 0 & 0 & \ldots & 0 & u_n v_1 - u_1 v_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & u_n v_{n-2} - u_{n-2} v_n \\ f(u_1 v_n - u_n v_1) & \ldots & f(u_{n-2} v_n - u_n v_{n-2}) & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad (9)$$

where $u = \sum u_i \partial_i$ and $v = \sum v_i \partial_i$. It is easily seen, by (9), that the eigenvalues of the skew-symmetric curvature operator vanish identically. This can be proved either by calculating the eigenvalues, or as a consequence of the fact that $R(\pi)$ is three-step nilpotent. Since all the eigenvalues of the skew-symmetric curvature operator are constant, we proved the following

Theorem 3. *All Egorov spaces are IP manifolds.*

We now turn our attention to ε-spaces. Using (1) and (8), it is easy to prove that the skew-symmetric curvature operator associated to any oriented non-degenerate 2-plane π is given

by

$$R(\pi) = \frac{\varepsilon}{\Xi} \begin{pmatrix} 0 & \cdots & 0 & u_1 v_{n-1} - v_1 u_{n-1} & 0 \\ 0 & \cdots & 0 & u_2 v_{n-1} - v_2 u_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & u_{n-2} v_{n-1} - v_{n-2} u_{n-1} & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ u_1 v_{n-1} - v_1 u_{n-1} & \cdots & u_{n-2} v_{n-1} - v_{n-2} u_{n-1} & 0 & 0 \end{pmatrix}, \quad (10)$$

where $\{u, v\}$, $u = \sum u_i \partial_i$, $v = \sum v_i \partial_i$, is a positively oriented orthonormal basis of π and $\Xi = |g(u, u)g(v, v) - g(u, v)^2|^{1/2}$. Now, (10) implies that the eigenvalues of the skew-symmetric curvature operator vanish identically (in particular, $R(\pi)$ is again three-step nilpotent). Hence, we have the following

Theorem 4. *All ε-spaces are IP manifolds.*

4   The Jacobi and Szabó operators

Let (M, g) be a pseudo-Riemannian manifold, p a point of M and $u \in T_p M$ a unit vector. Denote by $\gamma : r \mapsto \exp(ru)$ the geodesic with arc length r tangent to u at p. The *Jacobi operator* along γ is defined by $R_\gamma = R(\dot\gamma, \cdot)\dot\gamma$ (here, we are adopting the sign convention $R(u, v) = \nabla_{[u,v]} - [\nabla_u, \nabla_v]$ for the curvature tensor). In particular, $\mathcal{J}_u = R_\gamma(0)$ defines the Jacobi operator at u. The *Szabó operator* at u is defined by $\mathcal{S}_u := \mathcal{J}'_u := D_{\dot\gamma} R_\gamma(0)$. The role of the Jacobi operator in determining the geometry of (M, g) has been extensively studied. In particular, two natural generalizations of locally symmetric spaces were introduced in [2] in terms of the Jacobi operator. A manifold (M, g) is said to be

- a $\mathfrak{C}$-*space* if the eigenvalues of R_γ are constant along γ, for each geodesic γ of (M, g);

- a $\mathfrak{P}$-*space* if the Jacobi operator along any geodesic can be diagonalized by a parallel orthonormal frame along the geodesic.

As proved in [2, 6], a Riemannian or Lorentzian manifold (M, g) is locally symmetric if and only if it is both a $\mathfrak{C}$-space and a $\mathfrak{P}$-space. Moreover, at least in the analytic case, (M, g) is a $\mathfrak{P}$-space if and only if

$$[\mathcal{S}_u, \mathcal{J}_u] = 0 \qquad \text{for all } u, \tag{11}$$

that is, when the Jacobi and Szabó operators of all tangent vectors u commute. However, these results are no longer true in higher signatures, where examples exist which are $\mathfrak{C}$- and $\mathfrak{P}$-spaces simultaneously but not locally symmetric. Now, being symmetric, a ε-space is obviously both a $\mathfrak{C}$-space and a $\mathfrak{P}$-space. As concerns Egorov spaces, we calculated the Jacobi operator starting from (4), obtaining

$$\mathcal{J}_u = (n-2)\frac{(f')^2 - 2ff''}{4f^2} \begin{pmatrix} u_n^2 & 0 & \cdots & \cdots & 0 & -u_1 u_n \\ 0 & u_n^2 & 0 & \cdots & 0 & -u_2 u_n \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \cdots & u_n^2 & 0 & -u_{n-2} u_n \\ -fu_1 u_n & -fu_2 u_n & \cdots & -fu_{n-2} u_n & 0 & f\sum_{i \le n-2} u_i^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (12)$$

where $u = \sum u_i \partial_i$, while for the Szabó operator, from (3) and (12), we get

$$\mathcal{S}_X = \frac{1}{u_n} \left(\frac{(f')^2 - 2ff''}{4f^2} \right)' \mathcal{J}_u. \tag{13}$$

From (12) and (13) it follows at once that (11) holds for any tangent vector x. Thus, we proved the following

Theorem 5. *All Egorov spaces are $\mathfrak{P}$-spaces.*

We finish this section with the following remark on $\mathfrak{C}$-spaces. Since the eigenvalues of the Jacobi operators $\mathcal{J}_u$ are completely determined by the traces of the powers $\mathcal{J}_u^k$ for all k, a Lorentzian manifold is a $\mathfrak{C}$-space if and only if $\nabla_{\gamma'}$ trace $\mathcal{J}_\gamma^k = 0$ for all k and all geodesics γ.

The first of those conditions above ($\nabla_{\gamma'}$ trace $\mathcal{J}_\gamma = 0$) shows that the Ricci tensor is cyclic parallel (i.e., $(\nabla_X \text{Ric})(X, X) = 0$ for all vector fields X). Now, a straightforward calculation using the expressions in Section 2 shows that an Egorov space has cyclic parallel Ricci tensor if and only if it is locally symmetric (i.e., $(f')^2 - 2ff'' = 4f^2 C$ for some constant C).

5 Commuting curvature operators and semi-symmetry

Commuting properties of the curvature, Ricci and Jacobi operators have been intensively studied in the last years. For several of these properties, a complete description of the corresponding manifolds has not been obtained yet. A recent survey can be found in [3], and we can refer to [10] for the sistematic study of the three-dimensional Lorentzian case.

Let (M, g) be a pseudo-Riemannian manifold and denote by $\widehat{\text{Ric}}$ and $\mathcal{J}$ the Ricci operator and the Jacobi operator of (M, g), respectivey. The manifold is said to be

- *Jacobi–Ricci commuting* if $\mathcal{J}_u \cdot \widehat{\text{Ric}} = \widehat{\text{Ric}} \cdot \mathcal{J}_u$ for any tangent vector u;
- *curvature-Ricci commuting* if $R(u, v) \cdot \widehat{\text{Ric}} = \widehat{\text{Ric}} \cdot R(u, v)$ for all tangent vectors u, v;
- *curvature-curvature commuting* if $R(u, v) \cdot R(z, w) = R(z, w) \cdot R(u, v)$ for all tangent vectors u, v, z, w.

The class of Jacobi–Ricci commuting manifolds coincides with the one of curvature-Ricci commuting manifolds [14], which are also known in literature as *Ricci semi-symmetric spaces.* (We may refer to [17] for geometric interpretations and further references.) Other commuting curvature conditions, related to the *Weyl conformal curvature tensor* W, are also well known [3]. However, since both Egorov spaces and ε-spaces are conformally flat, their conformal curvature tensor identically vanishes and so, the commuting curvature conditions involving W are trivially satisfied.

For Egorov spaces $(\mathbb{R}^n, g_f)$, using (6) and (9), a straightforward calculation leads to a proof of the following

Theorem 6. *All Egorov spaces are curvature-Ricci commuting and curvature-curvature commuting.*

We briefly recall that a pseudo-Riemannian manifold (M, g) is said to be *semi-symmetric* if its curvature tensor R satisfies $R(u, v) \cdot R = 0$, for all tangent vectors u, v. Semi-symmetric spaces are a well known and natural generalization of locally symmetric spaces. A semi-symmetric space is Ricci semi-symmetric. The converse does not hold in general, but it is true for conformally flat manifolds, since their curvature is completely determined by the Ricci tensor. Hence, from Theorem 6 and (*iv*) of Theorem 1 we have at once the following

Theorem 7. *All Egorov spaces are semi-symmetric.*

Since ε-spaces are symmetric, they are semi-symmetric and thus Ricci semi-symmetric (i.e., curvature-Ricci commuting). Moreover, using (10), a direct calculation proves the following

Theorem 8. *All ε-spaces are curvature-curvature commuting.*

6 Recurrent curvature

A pseudo-Riemannian manifold (M, g) is said to be of *recurrent curvature* if the covariant derivative of its curvature tensor is linearly dependent of the curvature tensor itself. As Egorov spaces $(\mathbb{R}^n, g_f)$ are concerned, with respect to coordinate vector fields $\{\partial_i\}$, by (5) we have

$$\varrho = (n-2)\frac{(f')^2 - 2ff''}{4f^2} \begin{pmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \tag{14}$$

and so, (3) and (14) easily give

$$\nabla\varrho = (n-2)\left(\frac{(f')^2 - 2ff''}{4f^2}\right)' \begin{pmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Thus, the Ricci tensor of $(\mathbb{R}^n, g_f)$ is recurrent. Since $(\mathbb{R}^n, g_f)$ is conformally flat by (iv) of Theorem 1, we have at once the following

Theorem 9. *All Egorov spaces are of recurrent curvature.*

7 General conclusions

As we already mentioned in the Introduction, together with Lorentzian spaces of constant curvature and Lorentzian products of a space of constant curvature and a real line (or circle), Egorov spaces and ε-spaces complete the classification of Lorentzian manifolds in the third stratum, in all dimensions $n > 5$, $n \neq 7$. The socalled $\mathfrak{h}$-triple method, used in [21] to obtain the classification, does not apply for $n = 7$, even if Egorov and ε-spaces are defined in all dimensions greater than two. The classification above, together with the results of the previous sections, lead to the following.

> *For any dimension $n > 5$, $n \neq 7$, Lorentzian manifolds (M^n, g) in the third stratum are locally conformally flat and moreover, they are curvature-Ricci commuting (and thus semi-symmetric) and of recurrent curvature (and thus $\mathfrak{P}$-spaces).*

Some of the properties listed above are trivial for all Lorentzian manifolds in the third stratum except for Egorov spaces. However, they are interesting since they fix a common base of properties for manifolds belonging to the third stratum. As shown in [1], other properties, as symmetry and homogeneity, are not the common inheritance for these manifolds. We refer to Theorem 5.3 of [1] for other properties shared by all Lorentzian manifolds in the third stratum.

As concerns Lorentzian manifolds in the third stratum without Riemannian counterpart, we have in addition the following

> *Egorov-spaces and ε-spaces are Ivanov–Petrova and curvature-curvature commuting Walker manifolds.*

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